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Balance Index Set of Caterpillar and Lobster Graphs

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Abstract: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Consider the set $A = \{0, 1\}$. A labeling $f : V(G) \to A$ induces a partial edge labeling $f^* : E(G) \to A$ defined by $f^*(xy) = f(x)$, if and only if $f(x) = f(y)$ for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_{f^*}(i) = |\{e \in E(G) : f^*(e) = i\}|$. A labeling f of a graph G is said to be friendly if $|v_f(0) - v_f(1)| \leq 1$. A friendly labeling is balanced if $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$. The balance index set of the graph $G, BI(G)$, is defined as $\{|e_{f^*}(0) - e_{f^*}(1)| :$ the vertex labeling f is friendly$\}$. In this paper, we obtain the balance index set of caterpillar graphs and lobster graphs.

Key Words: Friendly labeling, Smarandache friendly labeling, partial edge labeling and balance index set.

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§1. Introduction

We begin with simple, finite, connected and undirected graph G=(V, E). Here the elements of set V and E are known as vertices and edges respectively with $|V| = p$ and $|E| = q$. For all other terminologies and notations we follow Harary [1].

Definition 1.1 *A path graph or linear graph is a tree with two or more vertices that contains only vertices of degree* 2 *and* 1*.*

Definition 1.2 *A caterpillar is a tree in which all the vertices are within distance* 1 *of a central path.*

Definition 1.3 *The graph $B_{l,m,k}$ is a tree obtained from a path of length k by attaching the stars $K_{1,l}$ and $K_{1,m}$ with its pendent vertices.*

Definition 1.4 *A coconut Tree $CT(m, l)$ is the graph obtained from the path P_m by appending l new pendent edges at an end vertex of P_m.*

Definition 1.5 *A lobster graph is a tree in which all the vertices are within distance* 2 *of a central path.*

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Definition 1.6 *A mapping $f : V(G) \to \{0,1\}$ is called friendly labeling of G if*

$$|v_f(0) - v_f(1)| \leq 1,$$

otherwise, a Smarandache friendly labeling of G, i.e., $|v_f(0) - v_f(1)| \geq 2$.

Lee, Liu and Tan [5] considered a new labeling problem of graph theory. A vertex labeling of G is a mapping f from $V(G)$ into the set $\{0,1\}$. For each vertex labeling f of G, a partial edge labeling f^* of G is defined in the following way.

For each edge uv in G,

$$f^*(uv) = \begin{cases} 0, & \text{if } f(u) = f(v) = 0 \\ 1, & \text{if } f(u) = f(v) = 1 \end{cases}$$

Note that if $f(u) \neq f(v)$, then the edge uv is not labeled by f^*. Thus f^* is a partial function from $E(G)$ into the set $\{0,1\}$. Let $v_f(0)$ and $v_f(1)$ denote the number of vertices of G that are labeled by 0 and 1 under the mapping f respectively. Likewise, let $e_{f^*}(0)$ and $e_{f^*}(1)$ denote the number of edges of G that are labeled by 0 and 1 under the induced partial function f^* respectively.

In [3] Kim, Lee, and Ng define the balance index set of a graph G as $\mathrm{BI}(G) = \{|e_{f^*}(0) - e_{f^*}(1)| : f^*$ runs over all friendly labelings f of $G\}$.

Example 1.7 Figure 1 shows a graph G with $BI(G) = \{0,1\}$.

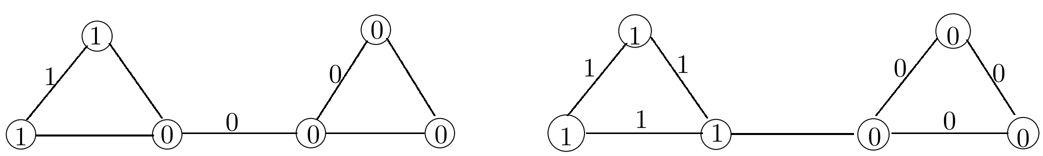

Figure 1 The friendly labelings of graph G with $BI(G) = \{0,1\}$.

For a graph with a vertex labeling f, we denote $e_{f^*}(X)$ to be the subset of $E(G)$ containing all the unlabeled edges. In [4] Kwong and Shiu developed an algebraic approach to attack the balance index set problems. It shows that the balance index set depends on the degree sequence of the graph.

Lemma 1.8([6]) *For any graph G,*

(1) $2e_{f^*}(0) + e_{f^*}(X) = \sum_{v\epsilon v(0)} deg(v)$;
(2) $2e_{f^*}(1) + e_{f^*}(X) = \sum_{v\epsilon v(1)} deg(v)$;
(3) $2\,|E(G)| = \sum_{v\epsilon v(G)} deg(v) = \sum_{v\epsilon v(0)} deg(v) + \sum_{v\epsilon v(1)} deg(v)$.

Corollary 1.9([6]) *For any friendly labeling f, the balance index is*

$$e_{f^*}(0) - e_{f^*}(1) = \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right).$$

More details of known results of graph labelings are given in Gallian [2].

In number theory and combinatorics, a partition of a positive integer n, also called an integer partition, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered to be the same partition; if order matters then the sum becomes a composition. For example, 4 can be partitioned in five distinct ways:

$$4+0, 3+1, 2+2, 2+1+1, 1+1+1+1.$$

Let G be any graph with p vertices. Partition of p in to $(p_0,\, p_1)$, where p_0 and p_1 are the number of vertices labeled by 0 and 1 respectively.

In [6] Lee, Su and Wang gave the results for balance index set of trees of diameter four. In this paper we obtain balance index set of caterpillar and lobster graphs of diameter n. To prove our result we are using Lemma 1.8 and Corollary 1.9.

§2. Balance Index Set of Caterpillar Graphs

Consider the caterpillar graph $CT(a_1, a_2, a_3, \cdots, a_{n-1})$, where a_i, i=1, 2, 3, $\cdots$,$n-1$ are the number of vertices adjacent to i^{th} spine vertices. We name $n-1$ vertices on the spine as $u_{a_1}, u_{a_2}, u_{a_3}, \cdots, u_{a_{n-1}}$. Thus for a caterpillar graph there are $(a_1 + a_2 + a_3 + \cdots + a_{n-1})$ number of pendant vertices. The degrees of $u_{a_1}, u_{a_2}, u_{a_3}, \cdots, u_{a_{n-1}}$ are $a_1 + 1, a_2 + 2, a_3 + 2, \cdots, a_{n-2} + 2, a_{n-1} + 1$ respectively. We also name non-spinal vertices adjacent to u_{a_1} by $u_{a_1,1}$, $u_{a_1,2}$, $u_{a_1,3}$, $\cdots$, u_{a_1,a_1}. Similarly we name non spinal vertices adjacent to u_{a_2}, u_{a_3}, u_{a_4}, $\cdots$, $u_{a_{n-1}}$.

Theorem 2.1 *For $CT(a_1, a_2, a_3, \cdots, a_{n-1})$ of order p and diameter n, the balance index is,*

$$e_{f^*}(0) - e_{f^*}(1) = \begin{cases} \left\{\dfrac{1}{2}\left(l + \sum\limits_{i=1}^{n-1}(-1)^{f(u_{a_i})}a_i\right)\right\}, & \text{if } p \text{ is even} \\ \\ \left\{\dfrac{1}{2}\left(l + 1 + \sum\limits_{i=1}^{n-1}(-1)^{f(u_{a_i})}a_i\right)\right\}, & \text{if } p \text{ is odd} \end{cases}$$

where

$$l = \begin{cases} n-2j-3, & \text{if } j = i, i-1, i-2, \text{ where } i = 2, 3, 4, \cdots, \left\lfloor \dfrac{n}{2} \right\rfloor \\ & \text{and j number of coefficients of } a_i \text{ are negative} \\ n-3, & \text{if } f(u_{a_i}) = 0 \text{ for all i or } f(u_{a_i}) = \begin{cases} 1, & \text{if } \ i = 1, n-1 \\ 0, & otherwise \end{cases} \end{cases}$$

Proof Consider the caterpillar graph $CT(a_1, a_2, a_3, \cdots, a_{n-1})$ of order p and diameter n.

Case 1. n is even.

Subcase 1.1 If $a_1+a_2+a_3+\cdots+a_{n-1}$ is odd, then the number of vertices of $CT(a_1, a_2, a_3,$

$\cdots, a_{n-1})$ is $a_1+a_2+a_3+\cdots+a_{n-1}+n-1$ which is even. Let $(a_1+a_2+a_3+\cdots+a_{n-1})+n-1 = 2M$. For a friendly labeling, M vertices are labeled 0 and remaining M vertices are labeled 1.

We first consider the case that $u_{a_1}, u_{a_2}, u_{a_3}, \cdots, u_{a_{n-1}}$ are all labeled 0, i.e. $n-1$ spine vertices are partitioned in to $(n-1, 0)$. Then $M-(n-1)$ pendant vertices are labeled 0 and M pendant vertices are labeled 1. Therefore by Corollary 1.9, we get

$$\begin{aligned}
e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\
&= \frac{1}{2}[M-(n-1)+(a_1+1)+(a_2+2)+(a_3+2)+\cdots+(a_{n-1}+1)-M] \\
&= \frac{1}{2}[a_1+a_2+a_3+\cdots+a_{n-1}+n-3].
\end{aligned}$$

If $n-1$ spine vertices are partitioned in to $(n-2, 1)$, then $M-(n-2)$ pendant vertices are labeled 0 and $M-1$ pendant vertices are labeled 1. Two possibilities arise.

(a) If the vertex u_{a_1} is labeled 1, then

$$\begin{aligned}
e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\
&= \frac{1}{2}[M-(n-2)-(a_1+1)+(a_2+2)+(a_3+2)+\cdots \\
&\quad +(a_{n-1}+1)-(M-1)] \\
&= \frac{1}{2}[-a_1+a_2+a_3+\cdots+a_{n-1}+n-3].
\end{aligned}$$

Similarly If $u_{a_{n-1}}$ is labeled 1, then

$$e_{f^*}(0) - e_{f^*}(1) = \frac{1}{2}[a_1+a_2+a_3+\cdots+a_{n-2}-a_{n-1}+n-3].$$

(b) If one vertex u_{a_i}, $i = 2, 3, 4, \cdots, n-2$ is labeled 1, then $M-(n-2)$ pendant vertices are labeled 0 and $M-1$ pendant vertices are labeled 1.

Therefore,

$$\begin{aligned}
e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\
&= \frac{1}{2}[M-(n-2)+(a_1+1)+(a_2+2)+(a_3+2)+\cdots-(a_i+2)+\cdots \\
&\quad +(a_{n-1}+1)-(M-1)] \\
&= \frac{1}{2}[a_1+a_2+a_3+\cdots+a_{i-1}-a_i+a_{i+1}+\cdots+a_{n-1}+n-5],
\end{aligned}$$

where $i = 2, 3, 4, \cdots, n-2$.

If $n-1$ spine vertices are partitioned in to $(n-1-i, i)$, where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$. Then $M-(n-1-i)$ pendant vertices are labeled 0 and $M-i$ pendant vertices are labeled 1. Three possibilities arise.

(a) If $f(u_{a_1}) = f(u_{a_{n-1}}) = 0$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v \epsilon v(0)} deg(v) - \sum_{v \epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[M - (n-1-i) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) + (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2i - 3], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and i coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

(b) If $f(u_{a_1}) = 0$ and $f(u_{a_{n-1}}) = 1$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v \epsilon v(0)} deg(v) - \sum_{v \epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[M - (n-1-i) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) - (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i - 1], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and $i-1$ coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

(c) If $f(u_{a_1}) = f(u_{a_{n-1}}) = 1$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v \epsilon v(0)} deg(v) - \sum_{v \epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[M - (n-1-i) - (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) - (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[-a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i + 1], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and $i-2$ coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

Subcase 1.2 If $a_1+a_2+a_3+\cdots+a_{n-1}$ is even, then the number of vertices of $CT(a_1, a_2, a_3, \cdots, a_{n-1})$ is $a_1+a_2+a_3+\cdots+a_{n-1}+n-1$ which is odd. Let $(a_1+a_2+a_3+\cdots+a_{n-1})+n-1 = 2M+1$.

For a friendly labeling, without loss of generality, there are $M+1$ vertices labeled 0 and M vertices labeled 1.

We first consider the case that $u_{a_1}, u_{a_2}, u_{a_3}, \cdots, u_{a_{n-1}}$ are all labeled 0, i.e. $n-1$ spine vertices are partitioned in to $(n-1, 0)$. Then $(M+1)-(n-1)$ pendant vertices are labeled 0 and M pendant vertices are labeled 1.

Therefore,

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-1) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-1}+1) - M] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2]. \end{aligned}$$

If $n-1$ spine vertices are partitioned in to $(n-2, 1)$, then $(M+1)-(n-2)$ pendant vertices are labeled 0 and $M-1$ pendant vertices are labeled 1. Two possibilities arise.

(a) If $f(u_{a_1}) = 1$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-2) - (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-1}+1) - (M-1)] \\ &= \frac{1}{2}[-a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2]. \end{aligned}$$

Similarly, if $f(u_{a_{n-1}}) = 1$ then

$$e_{f^*}(0) - e_{f^*}(1) = \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2].$$

(b) If one spine vertex of degree $a_i, i = 2, 3, 4, \cdots, n-2$ is labeled 1, then $M-(n-2)$ pendant vertices are labeled 0 and $M-1$ pendant vertices are labeled 1. Therefore,

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-2) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots - (a_i+2) + \cdots + (a_{n-1}+1) - (M-1)] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{i-1} - a_i + a_{i+1} + \cdots + a_{n-1} + n - 4], \end{aligned}$$

where $i = 2, 3, 4, \cdots, n-2$.

If $n-1$ spine vertices are partitioned in to $(n-1-i, i)$, where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$, then $(M+1)-(n-1-i)$ pendant vertices are labeled 0 and $M-i$ pendant vertices are labeled 1. Three possibilities arise.

(a) If $f(u_{a_1}) = f(u_{a_{n-1}}) = 0$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-1-i) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) + (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 2i - 2], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and i coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

(b) If $f(u_{a_1}) = 0$ and $f(u_{a_{n-1}}) = 1$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-1-i) + (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) - (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and $i-1$ coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

(c) If $f(u_{a_1}) = f(u_{a_{n-1}}) = 1$, then

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left(\sum_{v\epsilon v(0)} deg(v) - \sum_{v\epsilon v(1)} deg(v)\right) \\ &= \frac{1}{2}[(M+1) - (n-1-i) - (a_1+1) + (a_2+2) + (a_3+2) \\ &\quad + \cdots + (a_{n-2}+2) - (a_{n-1}+1) - (M-i)] \\ &= \frac{1}{2}[-a_1 + a_2 + a_3 + \cdots + a_{n-2} - a_{n-1} + n - 2i + 2], \end{aligned}$$

where $i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$ and $i-2$ coefficients out of $a_2, a_3, a_4, \cdots, a_{n-2}$ are negative.

Case 2. n is odd.

Subcase 2.1 If $(a_1 + a_2 + a_3 + \cdots + a_{n-1})$ is odd, then the number of vertices of $CT(a_1, a_2, a_3, \cdots, a_{n-1})$ is $a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 1$ which is odd. Therefore the proof is similar to Subcase 1.2.

Subcase 2.2 If $a_1+a_2+a_3+\cdots+a_{n-1}$ is even, then the number of vertices of $CT(a_1, a_2, a_3, \cdots, a_{n-1})$ is $a_1 + a_2 + a_3 + \cdots + a_{n-1} + n - 1$ which is even. Therefore the proof is similar to Subcase 1.1. □

Example 2.2 Figure 2 shows the caterpillar $CT(2, 1, 1, 2, 1)$ of diameter 6 and order 12 with balance index set $\{0, 1, 2, 3, 4, 5\}$.

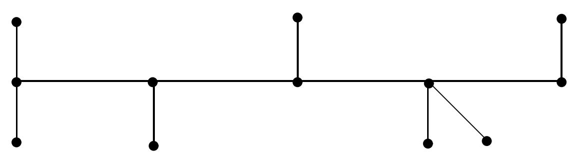

Figure 2 The caterpillar $CT(2, 1, 1, 2, 1)$ of diameter 6 and order 12.

Corollary 2.3 *The balance index set of the graph $B_{l,m,k}$,*

$$\begin{aligned} BI(B_{l,m,k}) &= \left\{ \left\lfloor \frac{l+m+k}{2} \right\rfloor, \left\lfloor \frac{|l-m+k|}{2} \right\rfloor, \left\lfloor \frac{|-l+m+k|}{2} \right\rfloor, \left\lfloor \frac{l+m+k-2}{2} \right\rfloor \right\} \bigcup \\ &\quad \left\{ \left\lfloor \frac{l+m+k-2i}{2} \right\rfloor, \left\lfloor \frac{|l-m+k-2i+2|}{2} \right\rfloor, \left\lfloor \frac{|-l-m+k-2i+4|}{2} \right\rfloor : \right. \\ &\quad \left. i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor \right\}. \end{aligned}$$

Proof The graph $B_{l,m,k}$ is a caterpillar $CT(l, 0, 0, \cdots, m)$ of diameter $k+2$. Therefore substituting $n = k+2$, $a_1 = l$, $a_{n-1} = m$ and $a_2 = a_3 = a_4 = \cdots = a_{n-2} = 0$ in the Theorem 2.1, we get

$$\begin{aligned} BI(B_{l,m,k}) &= \left\{ \left\lfloor \frac{l+m+k}{2} \right\rfloor, \left\lfloor \frac{|l-m+k|}{2} \right\rfloor, \left\lfloor \frac{|-l+m+k|}{2} \right\rfloor, \left\lfloor \frac{l+m+k-2}{2} \right\rfloor \right\} \\ &\quad \bigcup \left\{ \left\lfloor \frac{l+m+k-2i}{2} \right\rfloor, \left\lfloor \frac{|l-m+k-2i+2|}{2} \right\rfloor, \left\lfloor \frac{|-l-m+k-2i+4|}{2} \right\rfloor : \right. \\ &\quad \left. i = 2, 3, 4, \cdots, \left\lfloor \frac{n}{2} \right\rfloor \right\}. \end{aligned}$$

□

Example 2.4 Figure 3 shows the graph $B_{3,3,3}$ of diameter 5 and order 10 with balance index set $\{0, 1, 2, 3, 4\}$.

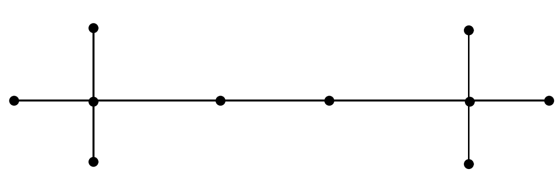

Figure 3 The graph $B_{3,3,3}$ of diameter 5 and order 10.

Corollary 2.5 *The balance index set of coconut tree $CT(m,l)$,*

$$\begin{aligned} BI(CT(m,l)) &= \left\{\left\lfloor\frac{l+m-2}{2}\right\rfloor, \left\lfloor\frac{|-l+m-2|}{2}\right\rfloor, \left\lfloor\frac{|l+m-4|}{2}\right\rfloor\right\}\bigcup \\ &\left\{\left\lfloor\frac{l+m-2i-2}{2}\right\rfloor, \left\lfloor\frac{|-l+m-2i|}{2}\right\rfloor, \left\lfloor\frac{|-l-m-2i+2|}{2}\right\rfloor : \right. \\ &\left. i=2,3,4,\cdots,\left\lfloor\frac{m}{2}\right\rfloor\right\}. \end{aligned}$$

Proof The coconut tree $CT(m,l)$ is a caterpillar graph $CT(0,0,0,\cdots,m)$ of diameter m. Therefore substituting $n=m$, $a_{n-1}=l$ and $a_1=a_2=a_3=\cdots=a_{n-2}=0$ in the Theorem 2.1, we get

$$\begin{aligned} BI(CT(m,l)) &= \left\{\left\lfloor\frac{l+m-2}{2}\right\rfloor, \left\lfloor\frac{|-l+m-2|}{2}\right\rfloor, \left\lfloor\frac{|l+m-4|}{2}\right\rfloor\right\}\bigcup \\ &\left\{\left\lfloor\frac{l+m-2i-2}{2}\right\rfloor, \left\lfloor\frac{|-l+m-2i|}{2}\right\rfloor, \left\lfloor\frac{|-l-m-2i+2|}{2}\right\rfloor : \right. \\ &\left. i=2,3,4,\cdots,\left\lfloor\frac{m}{2}\right\rfloor\right\}. \end{aligned}$$

□

Example 2.6 Figure 4 shows coconut tree of diameter 5 and order 9 with balance index set $\{0,1,2,3,5\}$.

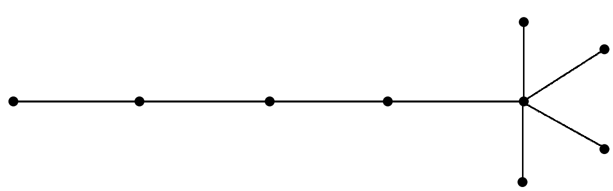

Figure 4 The coconut tree of diameter 5 and order 9.

§3. Balance Index Set of Lobster Graphs

In a caterpillar graph $CT(a_1,a_2,a_3,\cdots,a_{n-1})$, if $a_i \neq 0$ for $i=2,3,\cdots,n-2$, then we have $a_i, i=2,3,4,\cdots,n-2$ number of P_3 paths contained the vertex u_{a_i}. Since P_3 is of length 2, after adding more adjacent edges and vertices to the two end vertices of these paths, the new graph is a lobster graph of diameter n. We denote the new graph as

$$\begin{aligned} &CT(a_1,a_2,a_3,\cdots,a_{n-1})(u_{a_2}(t_{2,1},t_{2,2},t_{2,3},\cdots,t_{2,a_2}),(u_{a_3}(t_{3,1},t_{3,2},t_{3,3},\cdots,t_{3,a_3})), \\ &u_{a_4}(t_{4,1},t_{4,2},t_{4,3},\cdots,t_{4,a_4}),\cdots,u_{a_{n-2}}(t_{n-2,1},t_{n-2,2},t_{n-2,3},\cdots,t_{n-2,a_{n-2}})), \end{aligned}$$

where $t_{i,j}$ is the number of edges and vertices added to the vertex $u_{a_i,j}$, $i=2,3,4,\cdots,n-2$,

$j = 1, 2, 3, \cdots, a_i$. Here we have

$$a_1 + a_{n-1} + \sum_{i=2}^{n-2} \sum_{j=1}^{a_i} t_{i,j}$$

pendant vertices.

In order to write the results in an uniform manner we name this lobster graph as

$CT(d_1, d_2, d_3, \cdots, d_{n-2}, d_0)(u_{a_2}(d_{n-1}, d_n, d_{n+1}, \cdots, d_{d_2+n-2}),$
$u_{a_3}(d_{d_2+n-1}, d_{d_2+n}, d_{d_2+n+1}, \cdots, d_{d_2+d_3+n-2}),$
$u_{a_4}(d_{d_2+d_3+n-1}, d_{d_2+d_3+n}, d_{d_2+d_3+n+1}, \cdots, d_{d_2+d_3+d_4+n-2}), \cdots,$
$u_{a_{n-2}}(d_{d_2+d_3+\cdots+d_{n-3}+n-1}, d_{d_2+d_3+\cdots+d_{n-3}+n}, d_{d_2+d_3+\cdots+d_{n-3}+n+1}, \cdots, d_{d_2+d_3+\cdots+d_{n-2}+n-2}))$.

We also name $n-1$ spine vertices by $v_1, v_2, v_3, \cdots, v_{n-2}, v_0$, the vertices adjacent to v_2 by $v_{n-1}, v_n, v_{n+1}, \cdots, v_{d_2+n-2}$, adjacent to v_3 by v_{d_2+n-1}, v_{d_2+n}, v_{d_2+n+1}, $\cdots, v_{d_2+d_3+n-2}$, etc. and adjacent to v_{n-2} by $v_{d_2+d_3+d_4+\cdots+d_{n-3}+n-1}$, $v_{d_2+d_3+d_4+\cdots+d_{n-3}+n}$, $v_{d_2+d_3+d_4+\cdots+d_{n-3}+n+1}$, $\cdots, v_{d_2+d_3+d_4+\cdots+d_{n-2}+n-2}$.

Thus in this lobster, we have $d_0 + d_1 + \sum\limits_{i=n-1}^{m} d_i$ pendant vertices where $m = \sum\limits_{j=2}^{n-2} d_j + n - 2$, the degree of v_i for $n-1 \le i \le \sum\limits_{j=2}^{n-2} d_j + n - 2$ is $d_i + 1$ and the degree of v_k is $d_i + 2$ for $k = 2, 3, 4, \cdots, n-2$.

Theorem 3.1 *For a lobster graph of diameter n and order p, the balance index is*

$$e_{f^*}(0) - e_{f^*}(1) = \begin{cases} \frac{1}{2}\left[\sum\limits_{i=0}^{m}(-1)^{f(v_i)}d_i + \sum\limits_{i=2}^{n-2}(-1)^{f(v_i)}\right], & \text{if } p \text{ is even,} \\ \pm\frac{1}{2}\left[1 + \sum\limits_{i=0}^{m}(-1)^{f(v_i)}d_i + \sum\limits_{i=2}^{n-2}(-1)^{f(v_i)}\right], & \text{if } p \text{ is odd,} \end{cases}$$

where $m = \sum\limits_{j=2}^{n-2} d_j + n - 2$.

Proof Let G be a lobster graph of order p and diameter n.

Case 1. n is even.

Subcase 1.1 If $\sum\limits_{i=0}^{m} d_i$ is odd, then the number of vertices equal to $\sum\limits_{i=0}^{m} d_i + n - 1$ is even. Let it be $2M$. For a friendly labeling, there are M vertices labeled 0 and remaining M vertices labeled 1. We first consider the case that v_i for all i are labeled 0. Then there are $M - (n - 1) - \sum\limits_{i=2}^{n-2} d_i$ pendant vertices labeled 0 and remaining M pendant vertices labeled 1. Then by

Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left[M-(n-1)+2(n-3)+d_0+1+d_1+1+\sum_{i=n-1}^{m}(d_i+1)-M\right] \\ &= \frac{1}{2}\left[\sum_{i=n-1}^{m} d_i + \sum_{j=0}^{n-2} d_j + n-3\right] = \frac{1}{2}\left[\sum_{i=0}^{m} d_i + n-3\right]. \end{aligned}$$

Similarly we assume that there are k vertices among v_i for all $0 \le i \le \sum\limits_{j=2}^{n-2} d_j + n - 2$ labeled 0. Then there are $M-k$ pendant vertices labeled 0 and $M - \left[\sum\limits_{j=2}^{n-2} d_j + n - 1 - k\right]$ pendant vertices labeled 1. We define P to be the set containing all the 0-vertices among v_i for all $0 \le i \le \sum\limits_{j=2}^{n-2} d_j + n-2$. We also name N to be the set containing all the 1-vertices among v_i for all $0 \le i \le \sum\limits_{j=2}^{n-2} d_j + n - 2$. Then by Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left[M-k+\sum_{v\in P} deg(v) - \left(M - \left[\sum_{j=2}^{n-2} d_j + n-1-k\right] + \sum_{v \in N} deg(v)\right)\right] \\ &= \frac{1}{2}\left[M-k+\left(\sum_{v\in P}(deg(v)-1)+1\right) - M + \sum_{j=2}^{n-2} d_j\right] \\ &\quad + \left[n-1-k-\left(\sum_{v\in N}(deg(v)-1)+1\right)\right] \\ &= \frac{1}{2}\left[M-k+\left(\sum_{v\in P}(deg(v)-1)\right)+k-M+\sum_{j=2}^{n-2} d_j\right] \\ &\quad + \left[n-1-k-\left(\sum_{v\in N}(deg(v)-1)\right) - \left(\sum_{j=2}^{n-2} d_j + n-1-k\right)\right] \\ &= \frac{1}{2}\left[\sum_{v\in P}(deg(v)-1) - \sum_{v\in N}(deg(v)-1)\right]. \end{aligned}$$

Also note that

$$deg(v) - 1 = \begin{cases} d_i, & \text{if } i = 0,\, 1 \text{ and } n-3 \le i \le \sum\limits_{j=2}^{n-2} d_j + n-2 \\ d_i + 1, & \text{if } 2 \le i \le n-2 \end{cases}$$

Therefore

$$\begin{aligned} e_{f^*}(0) - e_{f^*}(1) &= \frac{1}{2}\left[(-1)^{f(v_0)} d_0 + (-1)^{f(v_1)} d_1 + \sum_{i=n-3}^{m} (-1)^{f(v_i)} d_i + \sum_{j=2}^{n-2} (-1)^{f(v_i)}(d_i+1)\right] \\ &= \frac{1}{2}\left[\sum_{i=0}^{m}(-1)^{f(v_i)} d_i + \sum_{j=2}^{n-2}(-1)^{f(v_i)}\right]. \end{aligned}$$

Subcase 1.2. If $\sum\limits_{i=0}^{m} d_i$ is even, then the number of vertices equal to $\sum\limits_{i=0}^{m} d_i + n - 1$ is odd. Let it be $2M+1$. For a friendly labeling, there are $M+1$ vertices labeled 0 and remaining M vertices labeled 1. We first consider the case that v_i for all i are labeled 0. Then there are $(M+1)-(n-1)-\sum\limits_{i=2}^{n-2} d_i$ pendant vertices labeled 0 and remaining M pendant vertices labeled 1. Then again by Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0)-e_{f^*}(1) &= \frac{1}{2}\left[(M+1)-(n-1)+2(n-3)+d_0+1+d_1+1+\sum_{i=n-1}^{m}(d_i+1)-M\right] \\ &= \frac{1}{2}\left[\sum_{i=n-1}^{m} d_i+\sum_{j=0}^{n-2} d_j+n-2\right]=\frac{1}{2}\left[\sum_{i=0}^{m} d_i+n-2\right]. \end{aligned}$$

Similarly we assume that there are k vertices among v_i for all $0\le i\le \sum\limits_{j=2}^{n-2} d_j+n-2$ labeled 0. Then there are $M+1-k$ pendant vertices labeled 0 and $M-\left[\sum\limits_{j=2}^{n-2} d_j+n-1-k\right]$ pendant vertices labeled 1. We define P to be the set containing all the 0-vertices among v_i for all $0\le i\le \sum\limits_{j=2}^{n-2} d_j+n-2$. We also name N to be the set containing all the 1-vertices among v_i for all $0\le i\le \sum\limits_{j=2}^{n-2} d_j+n-2$. Then by Corollary 1.9, we get

$$\begin{aligned} e_{f^*}(0)-e_{f^*}(1) &= \frac{1}{2}\left[\left(M+1-k+\sum_{v\in P} deg(v)\right)-\left(M-\left[\sum_{j=2}^{n-2} d_j+n-1-k\right]+\sum_{v\in N} deg(v)\right)\right] \\ &= \frac{1}{2}\left[M+1-k+\left(\sum_{v\in P}(deg(v)-1)+1\right)-M+\sum_{j=2}^{n-2} d_j\right. \\ &\quad \left.+\left[n-1-k-\left(\sum_{v\in N}(deg(v)-1)+1\right)\right]\right] \\ &= \frac{1}{2}\left[M+1-k+\left(\sum_{v\in P}(deg(v)-1)\right)+k-M+\sum_{j=2}^{n-2} d_j\right. \\ &\quad \left.+\left[n-1-k-\left(\sum_{v\in N}(deg(v)-1)\right)-\left(\sum_{j=2}^{n-2} d_j+n-1-k\right)\right]\right] \\ &= \frac{1}{2}\left[1+\sum_{v\in P}(deg(v)-1)-\sum_{v\in N}(deg(v)-1)\right]. \end{aligned}$$

Also note that

$$deg(v)-1=\begin{cases} d_i, & \text{if } i=0,\,1 \text{ and } n-3\le i\le \sum_{j=2}^{n-2} d_j+n-2 \\ d_i+1, & \text{if } 2\le i\le n-2 \end{cases}$$

Therefore,

$$\begin{aligned} e_{f^*}(0)-e_{f^*}(1) &= \frac{1}{2}\left[1+(-1)^{f(v_0)}d_0+(-1)^{f(v_1)}d_1+\sum_{i=n-3}^{m}(-1)^{f(v_i)}d_i+\sum_{j=2}^{n-2}(-1)^{f(v_i)}(d_i+1)\right] \\ &= \frac{1}{2}\left[1+\sum_{i=0}^{m}(-1)^{f(v_i)}d_i+\sum_{j=2}^{n-2}(-1)^{f(v_i)}\right]. \end{aligned}$$

When a friendly labeling with $v_f(1) > v_f(0)$, it produces the negative values of the above balance indexes. Therefore,

$$e_{f^*}(0) - e_{f^*}(1) = \pm\frac{1}{2}\left[1 + \sum_{i=0}^{m}(-1)^{f(v_i)}d_i + \sum_{j=2}^{n-2}(-1)^{f(v_i)}\right].$$

Case 2. n is odd.

Subcase 2.1 If $\sum\limits_{i=0}^{m} d_i$ is odd, then the number of vertices equal to $\left[\sum\limits_{i=0}^{m} d_i\right] + n - 1$ is odd and proof is similar to Subcase 1.2.

Subcase 2.2 If $\sum\limits_{i=0}^{m} d_i$ is even, then the number of vertices equal to $\left[\sum\limits_{i=0}^{m} d_i\right] + n - 1$ is even and proof is similar to Subcase 1.1.

Therefore, for a lobster graph of diameter n and order p, the balance index is

$$e_{f^*}(0) - e_{f^*}(1) = \begin{cases} \frac{1}{2}\left[\sum\limits_{i=0}^{m}(-1)^{f(v_i)}d_i + \sum\limits_{i=2}^{n-2}(-1)^{f(v_i)}\right], & \text{if } p \text{ is even} \\ \\ \pm\frac{1}{2}\left[1 + \sum\limits_{i=0}^{m}(-1)^{f(v_i)}d_i + \sum\limits_{i=2}^{n-2}(-1)^{f(v_i)}\right], & \text{if } p \text{ is odd} \end{cases} \qquad \square$$

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